

NON REDUCED PLANE CURVE SINGULARITIES WITH $b_1(F) = 0$

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ABSTRACT. If the first Betti number of the Milnor fibre of a plane curve singularity is zero, then the defining function is equivalent to x^r .

1. INTRODUCTION

What can be said about functions wth Milnor fibre F has the property $b_i(F) = 0$ for all $i \geq 1$? If F is connected then f is non-singular and equivalent to a linear function by A'Campo's trace formula. The remaining question: *What happens if F is non-connected ?* is only relevant for non-reduced plane curve singularities.

This question is related to a recent preprint [HM]. That preprint contains a statement about the so-called Bobadilla conjecture in case of plane curves. The invariant $\beta = 0$, used by Massey [Ma] should imply that the singular set of f is a smooth line.

In this short note we give a short topological proof of a stronger statement.

Proposition 1.1. *If the first Betti number of the Milnor fibre of a plane curve singularity is zero, then the defining function is equivalent to x^r .*

Corollary 1.2. *In the above case the singular set is a smooth line and the system of transversal singularities is trivial.*

2. NON-REDUCED PLANE CURVES

Non-isolated plane curve isolated singularities have been thoroughly studied by Rob Schrauwen in his dissertation [Sch1]. Main parts of it are published as [Sch2] and [Sch3]. The above Proposition 1.1 is an easy consequence of his work.

We can assume that $f = f_1^{m_1} \cdots f_r^{m_r}$ (partition in powers of reduced irreducible components).

Lemma 2.1. (Folklore) *Let $d = \gcd(m_1, \dots, m_r)$*

- (a.) *F has d components, each diffeomorphic to the Milnor fibre G of $g = g_1^{\frac{m_1}{d}} \cdots g_r^{\frac{m_r}{d}}$. The Milnor monodromy of f permutes these components,*
- (b.) *If $d = 1$ then F is connected.*

Proof. (a.) Since $f = g^d$ the fibre F consists of d copies of G .

(b.) We recall here the reasoning from [Sch1]. Deform the reduced factors f_i into \hat{f}_i such that the product $\hat{f}_1 \cdots \hat{f}_r = 0$ contains the maximal number of double points. This is called a network deformation by Schrauwen. The corresponding deformation \hat{f} of f near such a point has local equation are of the form $x^p y^q = 0$ (point of type $D[p, q]$).

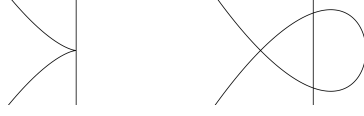


FIGURE 1. Deformation to maximal number of double points.

Near every branch $\hat{f}_i = 0$ the Milnor fibre is a m_i -sheeted covering of the zero-locus, except in the $D[p, q]$ -points. We construct the Milnor fibre F of f starting with $S = \sum m_i$ copies of the affine line \mathbb{A} . Cover the i^{th} branch with m_i copies of \mathbb{A} and delete $(p + q)$ small discs around the $D[p, q]$ -points. Glue in the holes $\gcd(p, q)$ small annuli (the Milnor fibres of $D[p, q]$). The resulting space is the Milnor fibre F of f .

Take a reference transversal fibre $F_1^\#$, which consists of m_1 cyclic ordered points. As soon as $f_1 = 0$ intersects $f_k = 0$ it connects the sheets of $f_1 = 0$ modulo m_k . Since $\gcd(m_1, \dots, m_r) = 1$ we connect all sheets. \square

Proof. (of proposition). If $b_1(F) = 0$, then also $b_1(G) = 0$. The Milnor monodromy has $\text{trace}(T_g) = 1$. According to A'Campo's observation [AC] g is regular: $g = x$. It follows that $f = x^r$. \square

3. RELATION TO BOBADILLA'S QUESTION

We consider first in any dimension $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ with a 1-dimensional singular set, see especially the 1991-paper [Si] for definitions, notations and statements. We focus on the group $H_n(F, F^\#)$ which occurs in two exact sequences on p. 468 of [Si]:

$$0 \rightarrow H_{n-1}^f(F) \rightarrow H_{n-1}(F^\#) \rightarrow H_n(F) \oplus H_{n-1}^t(F) \rightarrow 0$$

$$0 \rightarrow H_n(F) \rightarrow H_n(F, F^\#) \rightarrow H_{n-1}(F^\#) \rightarrow H_{n-1}(F) \rightarrow 0$$

Note that $H_n(F)$, $H_n(F, F^\#)$ and $H_{n-1}(F^\#)$ are free groups (all homologies here are taken over \mathbb{Z} , but also other coefficients are allowed).

Here $F^\#$ is the disjoint union of the transversal Milnor fibres $F_i^\#$, one for each irreducible branch of the 1-dimensional singular set.¹

From both sequences it follows that the β -invariant, introduced in [Ma] is nothing else than:

$$\dim H_n(F, F^\#) = b_n - b_{n-1} + \sum \mu_i^\# := \beta$$

It is immediately clear that $\beta \geq 0$ and that β is topological. This definition has as direct consequence:

Proposition 3.1.

$$\beta = 0 \Leftrightarrow H_n(F, \mathbb{Z}) = 0 \text{ and } H_{n-1}(F, \mathbb{Z}) = \mathbb{Z}^{\sum \mu_i}$$

The Bobadilla conjecture [Bo] was in [Ma] generalized as follows: *Does $\beta = 0$ imply that the singular set is smooth?* As consequence of our main Proposition 1.1 we have:

¹ $F^\#$ was originally denoted by F' . In the second second sequence a misprint n in the third term has been changed to $n - 1$.

Corollary 3.2. *In the curve case $\beta = 0$ implies that the singular set is smooth; and that the function is equivalent to x^r .*

NB. In [HM] the first part of this corollary was obtained with the help of Lê numbers.

From the definition $\beta = H_n(F, F^\natural)$ follow direct and short proofs of several statements in [Ma]. An other consequence from [Si] is the composition of surjections:

$$H_{n-1}(F^\natural) = \oplus \mathbb{Z}^{\mu_i} \twoheadrightarrow H_{n-1}(\partial_2 F) = \oplus \frac{\mathbb{Z}^{\mu_i}}{A_i - I} \twoheadrightarrow H_{n-1}(F)$$

From this follows:

Proposition 3.3. *If $\dim H_{n-1}(F) = \sum \mu_i$ (upper bound) then*

- a. $H_{n-1}(\partial_2 F)$ and $H_{n-1}(F)$ are free and isomorphic to $\mathbb{Z}^{\sum \mu_i}$.
- b. All transversal monodromies A_i are the identity.

The second part of [Ma] contains an elegant classification of $\beta = 1, 2$ with the help of the A'Campo trace formula. Also the reduction of the generalized Bobadilla conjecture to the (irreducible) Bobadilla conjecture. As final remark: The great work (the irreducible case) has still has to be done! Together with the Lê-conjecture this seems to be an important question in the theory of hypersurfaces 1-dimensional singular sets.

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